REMARKS ON DENJOY SETS

Osvald Demuth

Techn. Report No 30 September 1988

Charles University, Prague
Department of Computer Science
Malostranské náměstí 25
118 00 Praha 1, Czechoslovakia

LOGIC COLLOQUIUM '88
Osvald Demuth, Charles University, Malostranské nám. 25,
118 00 PRAHA 1, Czechoslovakia

Remarks on Denjoy sets.

Denjoy sets are sets of natural numbers corresponding to reals, which are interesting from the point of view of the theory of differentiation of constructive real functions. In my talk, some results concerning the structure of T- and tt-degrees containing Denjoy sets are presented. Methods of recursion theory and those of constructive mathematical analysis have been combined in the corresponding proofs. Let s, v, w, x, y and z be variables for natural numbers (NNs), A, B and C - variables for sets of NNs, σ and τ for binary strings (i.e. finite sequences of 0's and 1's), X and Y for reals.

We are supposed to have a fixed numbering of all binary strings (δ_x denotes the string with number x) such that, for any NNs x and y, if the string δ_x is either shorter than δ_y or preceding it lexicographically then $x < y$. For any set S of binary strings S^E denotes the class of all sets of NNs extending strings from S. We put $\langle A \rangle = \{\delta_x : x \in A\}$.

We use a standard notation for indexing of all partial recursive functions (PRFs), recursively enumerable (r.e.) sets of NNs and those of their relativizations (A-PRFs, A-r.e. sets) - φ_y, W_y, W_y^s, φ_y^A, W_y^A, $W_y^{A,s}$.

Binary expansions of reals give us a many-to-many correspondence between reals and sets of NNs. For any set A of NNs, we denote by r_A the sum of the series $\sum_{x \in A} 2^{-x-1}$ and, for any real X we denote by $Set(X)$ the infinite set B of NNs for which $X - r_B$ equals to an integer. Using reals to study sets of NNs we can restrict ourselves to reals from the closed unit interval $[0,1]$. A real X is said to be A-recursive if $Set(X) \leq_T A$ holds. In constructive mathematics in Markov's sense we study, among others, constructive reals (i.e. codes of $\emptyset$-recursive reals) and everywhere (i.e, for any constructive real) defined constructive functions of a real variable (briefly: constructive functions). Let us remember that any constructive function is an algorithm transforming equal constructive reals into equals one; and it is constructively continuous (i.e. continuous with an $\emptyset$-recursive function being a corresponding modulus of continuity) at any constructive real (cf. Kušner [1984]). Constructive functions constant on both $(-\infty, 0]$ and $[1, +\infty)$ we briefly call c-functions. It has turned out that, in constructive mathematical analysis, it is necessary to study also the behaviour of constructive functions in neighbourhoods of reals being non-($\emptyset$-recursive). In this connection, for any constructive function F and for any real X, we have defined the lower (classical) derivate and the upper (classical) derivate of F at X (and denoted them by $\underline{D}F(X)$ and $\overline{D}F(X)$, respectively) using, on account of the continuity of constructive functions at constructive reals, values of F at rational numbers only.

A constructive function is said to be constructively uniformly (or $\emptyset$-uniformly) continuous if there is an $\emptyset$-recursive function being a modulus of its uniform continuity. As has proved Zaslavskij, any monotone c-function is $\emptyset$-uniformly continuous.

For any constructive function F we denote by $R[F]$ a classical function of a real variable being maximal (as to domain) continuous (with respect to its domain) extension of F. Let us recall: For any c-function F, (F is classically uniformly continuous) $\Longleftrightarrow$ (F is $\emptyset'$-uniformly continuous) $\Longleftrightarrow$ ($R[F]$ is defined at any $\emptyset''$-recursive real) $\Longleftrightarrow$ ($R[F]$ is defined at any real) holds. Sacks [1963] introduced Lebesgue measure for classes of sets of NNs. Obviously, for any set S of binary strings, the class S^E is measurable (let $\mu(S^E)$ denotes its measure). The concept of $\emptyset$-measurability (i.e. constructive measurability) and its relativization have been introduced in constructive mathematics (Demuth [1969] and [1982A]) and in recursion theory (Demuth [1988A,B]). Here, we remind the way of introducing these concepts for special classes of sets of NNs only. A (total) function f is called a modulus of measurability of $\langle W_y^A \rangle^E$ if $\forall vw(f(v) \leq w \Rightarrow |\mu(\langle W_y^{A,f(v)} \rangle^E) - \mu(\langle W_y^{A,w} \rangle^E)| \leq 2^{-v})$ holds. The class $\langle W_y^A \rangle^E$ is said to be B-measurable if $A \leq_T B$

holds and there is a B-recursive function being a modulus of measurability of it. Let us note that the class $\langle W_y^A \rangle^E$ is necessarily A'-measurable but it cam be non-(A-measurable) (cf. Specker's example). In fact, this class is A-measurable if and only if its measure is an A-recursive real. The utility of this approach for recursion theory shows the following result.

<u>Lemma 1</u>. For any B-measurable class $\langle W_y^A \rangle^E$ and any binary string σ such that $\mu(\langle W_y^A \rangle^E) < \mu(\{\sigma\}^E)$, there is a B-recursive set C in the difference $\{\sigma\}^E \setminus \langle W_y^A \rangle^E$.

A class M of sets of NNs is said to be of <u>B-measure zero</u> if there are two B-recursive (or, equivalently, recursive) functions g and h of one variable such that, for any NN x, the class $\langle W^B_{g(x)} \rangle^E$ contains M, its measure is less than 2^{-x} and the function $\varphi^B_{h(x)}$ is a modulus of its measurability.

<u>Remark 2</u>. 1) The predicates $\mu(\langle W_y^s \rangle^E) > 2^{-v}$ of variables s, v and y and ($\langle W_y^s \rangle$ covers τ) (i.e. $\{\tau\}^E \subseteq \langle W_y^s \rangle^E$) of variables s, y and τ are, obviously, recursive and $\mu(\langle W_y^A \rangle^E) > 2^{-v}$ and ($\langle W_y^A \rangle$ covers τ) are A-recursively enumerable.

2) Let k be a recursive function of two variables fulfilling $k(x,s) = \mu y$ (the length of δ_y is x and either δ_y is not covered by $\langle W_x^s \rangle$ or δ_y is a string of x 1's). Thus, for any NN x, the recursive sequence $\{k(x,s)\}_s$ is non-decreasing and contains NNs from $\{v : 2^x - 1 \leq v \leq 2^{x+1} - 2\}$ only and, consequently, $\mathrm{Card}(\{s : k(x,s) \neq k(x,s+1)\}) < 2^{-x}$. Hence, $\lambda x \lim_{s\to\infty} k(x,s)$ is an $\emptyset'$-recursive function, $\mu(\langle\{\lim_{s\to\infty} k(x,s)\}\rangle^E) = 2^{-x}$ for any NN x and the class $\bigcap_{v=0}^{+\infty} \bigcup_{x>v} \langle\{\lim_{s\to\infty} k(x,s)\}\rangle^E$ is of $\emptyset'$-measure zero. It can be easily shown that this class contains any set of NNs which T-degree is hyperimmune-free.

A set S of binary strings is called a <u>covering</u> if it is r.e. and the class S^E contains all recursive sets. A set A of NNs is called a <u>semigeneric set</u> (Demuth [1987A]) if it is non-recursive and

$A \in S^E$ holds for any covering S. Let us notice that semigenericity is a generalization of weak 1-genericity introduced and studied by Kurtz [1983] which has proved that a T-degree contains a weakly 1-generic set if and only if it contains a hyperimmune set. Let us notice that the class of all weakly 1-generic sets is of $\emptyset$-measure zero. Results on semigeneric sets can be found in Demuth [1987A] and Demuth, Kučera [1987]. Here we remind a few of them only.

Remark 3. 1) Any set of NNs tt-reducible to a semigeneric set is either semigeneric or recursive. Thus, the class of all non-recursive non-semigeneric T-degrees (or, tt-degrees) is closed upwards.

2) Any hyperimmune set is semigeneric, but, there are semigeneric sets in some hyperimmune-free (T-) degrees. The class of all semigeneric T-degrees (or, tt-degrees) is not closed upwards.

Definition. Let z be a NN. A set A of NNs is called

(a) a Denjoy set if there is no constructive function F such that $\underline{D}F(r_A) = +\infty$ holds;

(b) an AP-set if there is a recursive function f such that $A \in \langle W_{f(x)} \rangle^E$ and $\mu(\langle W_{f(x)} \rangle^E) \leq 2^{-x}$ hold for any NN x (the term "effectively approximable by Σ^0_1 classes in measure" was introduced by Kučera [1985]);

(c) a NAP-set if it is not an AP-set;

(d) a z-WAP-set (z-weakly approximable ...) if φ_z is a total function and there is a recursive function g of two variables such that

$$(1) \quad \forall x \left(\mathrm{Card}(\{ y : g(x,y) \neq g(x,y+1) \}) \leq \varphi_z(x) \ \& \ \mu(\langle W_{\lim_{y \to \infty} g(x,y)} \rangle^E) \leq 2^{-x} \right)$$

and

$$(2) \quad \mathrm{Card}(\{ x : A \in \langle W_{\lim_{y\to\infty} g(x,y)} \rangle^E \}) = +\infty ;$$

(e) a WAP-set if it is a y-WAP-set for some NN y;

a NWAP-set if it is not a WAP-set.

Let us notice that the classes of arithmetical reals corresponding to these types of sets were introduced in Demuth [1982A]

<u>Remark 4.</u> 1) Any AP-set is, obviously, a WAP-set and, according to Demuth [1975B], any non-Denjoy set is necessarily an AP-set. Thus, any NWAP-set is a NAP-set any any NAP-set is a Denjoy set.

2) There is a recursive function e such that, for any NN x, the set $\langle W_{e(x)} \rangle$ is a covering, the class $\langle W_{e(x)} \rangle^E$ contains all AP-sets and its measure is less than 2^{-x} (Marti-Löf [1970], Demuth [1975A]). Hence, any semigeneric set is an AP-set and the class of all AP-sets is a Π_2^0 class of $\emptyset'$-measure zero.

3) For any class M of sets of NNs of $\emptyset$-measure zero we can find an increasing constructive function G fulfilling $\underline{D}G(r_A) = +\infty$ for any set A from M. On the other hand, there are sets of NNs being both Denjoy sets and AP-sets among semigeneric sets and among non-semigeneric sets, too (Demuth [1987B]). It can be shown that such sets are even in r.e. tt-degrees.

4) The class of all WAP-sets is of $\emptyset'$-measure zero (Demuth [1982A]). If we replace the condition (2) in the definition of z-WAP-sets by

$$(3) \qquad \forall x \left(A \in \langle W_{\lim_{y \to \infty} g(x,y)} \rangle^E \right) ,$$

we obtain another class of sets. We will call these sets z-WAP*-sets and define WAP*-sets in the way analogical to that used in (e). According to Demuth [1982A] there is a recursive function f of two variables such that the class $\bigcup_{x=0}^{+\infty} \bigcap_{y=0}^{+\infty} \langle W_{f(x,y)} \rangle^E$ is of $\emptyset'$-measure zero and contains all WAP*-sets. It has been shown in the paper that for each class of the just described type we can construct an $\emptyset'$-recursive WAP-set being not in it. The (used) method is as follows: We construct a recursive function g of two variables such that, for any NN v, the sequence $\{g(v,z)\}_z$ is non-decreasing and containing NNs from $\{s : 2^v - 1 \leq s \leq 2^{v+1} - 2\}$ only, i.e. the length of $\delta_{g(v,z)}$ is v for any NN z, and, in addition, if v is positive and, for the finite increasing sequence $\{n_i\}_{i=0}^m$ of NNs satisfying $v = \sum_{i=0}^m 2^{n_i}$, the measure of $\emptyset'$-measurable classes $\bigcap_{j=0}^{n_i} \langle W_{f(i,j)} \rangle^E$ is sufficiently small for $0 \leq i \leq m$ then the mea-

sure of the intersection of the classes $\bigcup_{i=0}^{m} \bigcap_{j=0}^{n_i} \langle W_{f(i,j)} \rangle^E$ and $\{\delta_{\lim_{z\to\infty} g(v,z)}\}^E$ is small with respect to $\mu(\{\delta_{\lim_{z\to\infty} g(v,z)}\}^E)$.
Then we find an $\emptyset'$-recursive function h of one variable such that $\{h(s)\}_s$ is an increasing sequence of NNs fulfilling the described additional conditions and such that the binary string with number $\lim_{z\to\infty} g(h(s+1),z)$ is an extension of the string with number $\lim_{z\to\infty} g(h(s),z)$ for any s. This is possible because of $\emptyset'$-decidability of the corresponding properties. This example shows that we can obtain WAP-sets with special properties using standard tree-methods and oracle constructions. We have seen that the replacing of the condition (2) by (3) in the definition (d) would change the situation strongly.

5) The preceding points show that any of the classes - of all semigeneric sets, all non-Denjoy sets, all AP-sets and all WAP-sets - is of $\emptyset'$-measure zero and cannot be of $\emptyset$-measure zero. Thus, $\emptyset'$-almost any set of NNs is a NWAP-set. The class of all non-Denjoy sets is a Σ^0_3 class being neither a $\Pi_2^{0,\emptyset^{(n)}}$ class for some NN n nor a Π^0_3 class. The class of all WAP-sets is both a $\Sigma_3^{0,\emptyset'}$ class and a Σ^0_4 class being neither a Π^0_4 class nor a $\Pi_3^{0,\emptyset^{(n)}}$ class for some NN n (Demuth [1982B], [1987B] and [1988B]).

It is useful to divide the class of all Denjoy sets into three classes: the class $\mathcal{D}_1$ of all sets being both Denjoy sets and AP-sets; the class $\mathcal{D}_2$ of all sets being both NAP-sets and WAP-sets and the class $\mathcal{D}_3$ - class of all NWAP-sets.

Properties of reals corresponding to sets of the introduced types have been studied in constructive mathematics.

<u>Theorem 5</u>. 1) For any $\emptyset$-uniformly continuous c-function F and any Denjoy set A the Denjoy relations for Dini derivatives - see Thomson [1985] - are valid for F at r_A (Demuth [1980]), in particular,

$$(4) \qquad \underline{D}F(r_A) = -\infty \;\&\; \overline{D}F(r_A) = +\infty \;\vee\; -\infty < \underline{D}F(r_A) = \overline{D}F(r_A) < +\infty$$

holds. Obviously, only Denjoy sets can have the described property (Demuth [1975B]). Consequently, any monotone c-function has a finite classical derivative (briefly: derivative) at r_A for any Denjoy set A of NNs.

2) There are a c-function F of classically bounded variation and a set A of NNs being both a Denjoy set and an AP-set such that (4) is not valid. Let us notice that any c-function of classically bounded variation is necessarily $\emptyset'$-uniformly continuous.

3) Any c-function of classically bounded variation has a derivative at r_A for any NAP-set A (Demuth [1975B]).

4) There are an $\emptyset'$-uniformly continuous c-function F and a NAP-set A such that (4) is not valid (Demuth [1976]). The following point shows that (such an) A is necessarily a WAP-set.

5) For any c-function F and any NWAP-set A, (4) is valid (Demuth [1983]).

We can use $\emptyset$-uniformly continuous c-functions for the study of tt-reducibility of sets of NNs (Demuth [1988A] and [1988B]).

<u>Theorem 6.</u> There is a class $\mathcal{E}$ of sets of NNs of $\emptyset$-measure zero with the following properties:

1) For any sets A and B of NNs a)

(5) $\quad A \leq_{tt} B$

implies

(6) (there is an $\emptyset$-uniformly continuous c-function F such that $R[F](r_B) = r_A$)

whenever B is not in $\mathcal{E}$;

b) (6) implies (5) whenever A is not in $\mathcal{E}$.

2) The class $\mathcal{E}$ contains no bi-infinite set being either recursive or of the type (C join C). As a class of $\emptyset$-measure zero $\mathcal{E}$ contains no Denjoy set.

<u>Theorem 7.</u> For any $\emptyset$-uniformly continuous c-function F we can construct a non-decreasing (and, thus, $\emptyset$-uniformly continuous) c-function G such that for any non-recursive set A , where r_A is in

the range of $R[F]$, there is a unique set C fulfilling the following conditions (a) - (c) .

(a) $R[G](r_C) = r_A$ and $C \equiv_T A$ hold.

(b) r_C is the measure of any of the following A-measurable classes $\{B : R[F](r_B) \leq r_A\}$ and $\{B : R[F](r_B) < r_A\}$.

(c) Let B be a set of NNs fulfilling $R[F](r_B) = r_A$. Then (C is an AP-set) $\Rightarrow$ (B is an AP-set) holds and, for any NN z , (C is a z-WAP-set) $\Rightarrow$ (B is a z-WAP-set) is valid.

Corollary 8. Let A and B be sets of NNs satisfying $B \notin \mathcal{E}$ and $\emptyset <_{tt} A \leq_{tt} B$. Then there is a set C of NNs such that $A \leq_{tt} C \leq_T A$, (B is a NAP-set) $\Rightarrow$ (C is a NAP-set) and (B is a NWAP-set) $\Rightarrow$ (C is a NWAP-set) hold.

Proof. It is sufficient to use $A \equiv_{tt}$ (A join A) and Theorems 6 and 7.

Now, we list some results about the structure of T- and tt-degrees containing Denjoy sets.

Remark 9. 1) If sets A and B are T-comparable then the set (A join B) is an AP-set (Kučera [1985]); if the sets A and B are tt-comparable then the set (A join B) is in a class of $\emptyset$-measure zero (Demuth [1987A]) and, consequently, it is not a Denjoy set.

2) If at least one of the sets A and B is a non-Denjoy set (or, as the case may be, an AP-set, or, a WAP-set) then the set (A join B) has this property, too.

3) According to 1) and 2), no minimal T-degree contains a NAP-set and no minimal tt-degree contains a Denjoy set.

4) There is a hyperimmune-free T-degree containing a NAP-set / The part 2) of Remark 4 and Theorem 2.4 in Jockusch and Soare [1972]).

Theorem 10 (Demuth [1987A]). Under any hyperimmune-free T-degree containing a NAP-set there is no minimal T-degree.

Proof. Let A be a non-recursive set and B a NAP-set such that $A \leq_T B$ holds and there is no hyperimmune set T-equivalent to B.

According to the proof of Theorem 3.12 from Odifreddi [1981], we obtain $\emptyset' <_{tt} A \leq_{tt} B$ and, by Corollary 8 and the part 3 of Remark 9, the T-degree of the set A is not a minimal one.

Remark 11. According to Remark 2, any set contained in a hyperimmune-free T-degree is a WAP-set.

Theorem 12 (Demuth [1988B]). Let A be both a NAP-set and a WAP-set. Then any set B such that $A \leq_T B$ holds is necessarily a WAP-set.

Remark 13. There is a set being both a NAP-set and a WAP-set which is contained in a r.e. tt-degree. By this and Theorem 12, any set B satisfying $\emptyset' \leq_T B$ is a WAP-set.

The class $\mathcal{D}_1$ /i.e. the class of all sets being both Denjoy sets and AP-sets/:

1) Any set A from $\mathcal{D}_1$ fulfils $\emptyset'' \leq_T A'$, i.e there is an A-recursive function majorizing any recursive function almost everywhere, where a.e. = for any sufficiently large NN. Hence, there is a hyperimmune set T-equivalent to A (Martin).

2) Let A be a set of NNs satisfying $\emptyset'' \leq_T A'$. Then we have the following:

a) There is a semigeneric set in $\mathcal{D}_1$ being T-reducible to A;

b) If A is $\emptyset'$-recursive then there is a non-semigeneric set in $\mathcal{D}_1$ being T-equivalent to A.

c) If the set A is an $\emptyset^{(x+1)}$-recursive set for some NN x then there is a set in $\mathcal{D}_1$ being T-equivalent to A.

Theorem 14. There is a minimal T-degree containing a semigeneric set being both a Denjoy set and an AP-set /cf. the part 3 of Remark 9/.

Proof. According to Epstein [1979], Cooper has constructed a set A which T-degree is a minimal one and $\emptyset'' \equiv_T A'$ is fulfilled It suffices to use 2a).

Example 15. There is a set C of NNs such that $C' \equiv_T \emptyset''$ holds and any set T-reducible to C is either recursive or semigeneric.

The class $\mathcal{D}_2$ /i.e. the class of all sets being both NAP-sets and WAP-sets/.

1) For any set A fulfilling $\emptyset' \leq_T A$ there is a NAP-set T-equivalent to it (Kučera [1985]). By Remark 13, such a NAP-set is necessarily in $\mathcal{D}_2$.

2) There are NAP-sets in hyperimmune-free T-degrees (Remark 9). According to Remark 11, they must be in $\mathcal{D}_2$.

Example 16 (Demuth [1987A]). There are a NAP-set A and a set B such that $A \leq_{tt} B$ and the T-degree of B is both hyperimmune-free and NAP-free. This T-degree must be Denjoy free, too. Indeed, as we have noticed, any member of $\mathcal{D}_1$ is T-equivalent to a hyperimmune set.

The class $\mathcal{D}_3$ - class od all NWAP-sets.

The class of all WAP-sets is of $\emptyset'$-measure zero (part 5 of Remark 4). Thus, $\emptyset'$-almost any set of NNs is in $\mathcal{D}_3$ (cf. Lemma 1). In particular, most of (in the sense of measure) $\emptyset'$-recursive sets are NWAP-sets.

Theorem 17 (Demuth [1988B]). 1) For any NWAP-set A we have $(A \text{ join } \emptyset') \equiv_T A'$.

2) For any set C of NNs satisfying $\emptyset' \leq_T C$ there is a NWAP-set A such that $A' \equiv_T (A \text{ join } \emptyset') \equiv_T C$ is valid.

Now, we turn to the question of the reducibility of members of different classes among those introduced by us.

T-reducibility. 1) The T-degree of ~~a~~ set A, satisfying $\emptyset' \leq_T A$ and $\exists x \, (A \leq_T \emptyset^{(x)})$, contains members of $\mathcal{D}_1$ and of $\mathcal{D}_2$, too (see results about these classes).

2) Let C be a set of NNs satisfying $\emptyset'' \leq_T C$ and $C \leq_T \emptyset^{(x)}$ for some NN x. Then, as we know, there are a set A from $\mathcal{D}_3$ fulfilling $A' \equiv_T C$ and, consequently, a set B from $\mathcal{D}_1$ T-equivalent to A.

3) No member of $\mathcal{D}_3$ is T-reducible to a member of $\mathcal{D}_2$ (Theorem 12)

4) As we have already noticed, there are many members of $\mathcal{D}_3$ being T-reducible to $\emptyset'$ and the T-degree of this set contains elements from $\mathcal{D}_2$ (see the part 1).

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tt-reducibility. 1) No member of $\mathcal{D}_1$ is tt-reducible to a NAP-set (i.e. to a member of either $\mathcal{D}_2$ or $\mathcal{D}_3$) and no member of $\mathcal{D}_2$ is tt-reducible to a member of $\mathcal{D}_3$.

Indeed, let B be a NAP-set and let $\emptyset <_{tt} A \leq_{tt} B$. By Theorems 6 and 7, there are a non-decreasing ($\emptyset$-uniformly continuous) c-function G and a NAP-set C such that $R[G](r_C) = r_A$ and (B is a NWAP-set) $\Rightarrow$ (C is a NWAP-set) hold. We use Theorem 5. The c-function G has a non-negative classical derivative at r_C. If the derivative is positive then by results from Demuth [1983] C and A must be in the same of our three classes. If the derivative is zero then, by Lemma 3 from Demuth [1980], A cannot be a Denjoy set.

2) There are members of $\mathcal{D}_3$ being tt-reducible to members of $\mathcal{D}_2$ (resp. of $\mathcal{D}_1$) and members of $\mathcal{D}_2$ tt-reducible to members of $\mathcal{D}_1$ (cf. Demuth [1988B]). The strongly oscillating c-function F constructed in the proof of Theorem 18 of the named paper can be used in the proof of all parts of this statement.

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